研究会報告

On Algebraic Knots I

— Computatability of Their Jones Polynomials —

Yuanan Diao[†1], Claus Ernst[‡2] and Uta Ziegler[§3]
[†] Department of Mathematics and Statistics
University of North Carolina at Charlotte
Charlotte, NC 28223, USA
[‡]Department of Mathematics
[§]Department of Computer Science
Western Kentucky University
Bowling Green, KY 42101, USA

Abstract: We prove that the Jones polynomial of any Conway algebraic link diagram with n crossings can be computed in $O(n^2)$ time. In particular, the Jones polynomial of any Montesinos link and two-bridge knot or link with minimum crossing number n can be computed in $O(n^2)$ time.

1 Conway algebraic knots

A knot K is called a *Conway algebraic knot* (or just algebraic knot for short) if it admits a diagram D in S^2 such that there exists a set of disjoint simple closed curves $C_1, C_2, \ldots, C_n$ such that each C_j intersects D transversely in exactly four non-crossing points of D and the regions bounded by each C_j are of the two types shown in Figure 1. Each such simple closed curve C_j is referred to as a Conway circle. Notice that the type B regions bounded by the Conway circles shown in the figure do not contain any crossings of D.

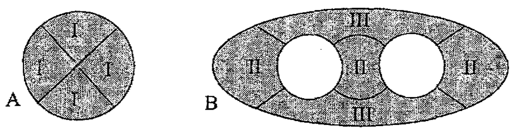

Figure 1: A. A Conway region containing a single Conway circle with a single crossing from the knot diagram in it. B. A Conway region bounded by three Conway circles without any crossings from the knot diagram.

In general, it is not known which algebraic knots admit a minimal crossing diagram that can be decomposed by Conway circles as described above. There exist algebraic knots whose minimal diagrams are not algebraic diagrams. The class of algebraic knots is very large: it contains all two-bridge knots and all Montesinos knots, as well as many other knots. For a more detailed discussion on algebraic knots, one may refer to [2].

[1]E-mail: ydiao@uncc.edu
[2]E-mail: claus.ernst@wku.edu
[3]E-mail: uta.ziegler@wku.edu

2 Algebraic diagrams and tangle replacement operations

Given a simple link diagram D_0 and a 2-string tangle Γ_1, a *tangle replacement operation* on D_0 using Γ_1 is to replace a crossing in D_0 by Γ_1. See Figure 2 for an example. This results in a diagram D_1. The tangle replacement operation may be repeated on D_1 using a different 2-string tangle Γ_2 to obtain yet another new knot diagram D_2. If this process is repeated m times using tangles Γ_1, Γ_2, ..., Γ_m, we obtain a knot diagram D_m. Let $|\Gamma_j|$ be the number of crossings in Γ_j and let $d = \max\{|\Gamma_1|, |\Gamma_2|, \cdots, |\Gamma_m|, |D_0|\}$. We say that D_m is obtained by m tangle replacement operations (using the tangles Γ_1, Γ_2, ..., Γ_m) with *tangle depth* d. Of course, the tangle depth of the knot diagram D_m depends on the particular choice of the tangle replacement operation sequence which may not be unique.

Figure 2: Tangle replacement operation: replacing a single crossing by a tangle.

By induction on the number of crossings in a diagram one can prove the following theorem.

Theorem 2.1 *If D is an algebraic knot diagram with n crossings, then it can be obtained by a sequence of tangle replacement operations with tangle depth 2.*

3 Face graphs of algebraic knot diagrams

Figure 3 shows an example of how to obtain a (signed) face graph of a knot projection: shade the regions of D is a checkerboard fashion and each shaded region corresponds to a vertex in the face graph F_D and a crossing between two shaded regions becomes an edge in F_D connecting the two vertices corresponding to these regions. The signs of the edges in F_D are determined by the relation of the shaded regions at the crossing in D as shown in Figure 3.

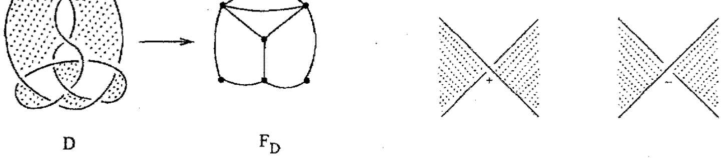

Figure 3: A projection of the knot 9_{45} and its corresponding face graph.

Replacing a crossing in D by tangle Γ corresponds to replacing an edge in F_D by a plane graph N, see Figure 4. N is obtaind from a facegraph generated by shading a knot or link diagram D' that contains Γ and an addtional crossing c. The extra crossing c corresponds to an edge with label e and N equals to the face graph of $F_{D'} - e$. The operation that replaces an edge e in F_D by a plane graph N as shown in the figure is called a (single edge) *tensor product* of F_D and N and is denoted by $F_D \otimes_e N$ (or simply $F_D \otimes N$). Using Theorem 2.1, we can then prove the following theorem.

Theorem 3.1 *Let D be an algebraic knot diagram with n crossings, then there exists a sequence of (connected) plane graphs D_0, N_1, ..., N_{n-2} such that each of these graphs contains exactly two edges and the face graph $F_D = (\cdots((D_0 \otimes N_1) \otimes N_2) \otimes \cdots) \otimes N_{n-2}$.*

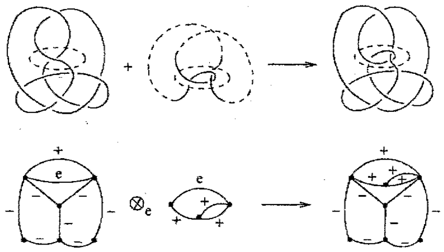

Figure 4: A knot diagram before and after a tangle replacement and the corresponding face graphs.

4 Tutte polynomials of colored graphs

A colored graph is a graph G in which each edge is assigned a color from a color set Λ. If $\Lambda = \{+, -\}$, then G is called a signed graph. Four variables x_λ, y_λ, X_λ and Y_λ are associated with each color $\lambda \in \Lambda$. Let e_λ be an edge in G with color λ, then the (colored) Tutte polynomial of G can be defined recursively by

$$T(G) = \begin{cases} Y_\lambda T(G \setminus e), & \textit{if } \mathrm{e} \textit{ is a loop} \\ X_\lambda T(G/e), & \textit{if } \mathrm{e} \textit{ is a bridge,} \\ y_\lambda T(G \setminus e) + x_\lambda T(G/e), & \textit{otherwise.} \end{cases}$$

It is a well known result of Bollobás and Riordan [1] that $T(G)$ is well-defined (that is, it is independent of the order of the edges used in the recursions) as long as the color variables satisfy the following conditions:

$$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} = \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} \text{ and } \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} = \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}.$$

For the sake of convenience, sometimes we just use the name of an edge as its color.

Theorem 4.1 *[5] Let M and N be two colored graphs such that the color variables involved satisfy the above conditions. Furthermore, assume that N is a connected graph with exactly two edges (with colors c_1 and c_2), then $T(M \otimes_e N)$ can be obtained from $T(M)$ by the following color variable assignments to X_e, Y_e, x_e and y_e:*

Case 1. N consists of a two edge path:

$$X_{c_1}X_{c_2} \mapsto X_e \quad x_{c_1}x_{c_2} \mapsto x_e \quad y_{c_1}X_{c_2} + x_{c_1}Y_{c_2} \mapsto Y_e \quad y_{c_1}X_{c_2} + x_{c_1}y_{c_2} \mapsto y_e.$$

Case 2. N is a cycle consisting of two vertices and two edges:

$$y_{c_1}X_{c_2} + x_{c_1}Y_{c_2} \mapsto X_e \quad y_{c_1}x_{c_2} + x_{c_1}Y_{c_2} \mapsto x_e \quad Y_{c_1}Y_{c_2} \mapsto Y_e \quad y_{c_1}y_{c_2} \mapsto y_e.$$

5 The Jones polynomials of algebraic knots

For knots with a large crossing number, the computation of their knot invariants can be very difficult. For example, the computation of the Jones polynomial of a link is known to be NP-hard [6]. This prevents the computation of the Jones polynomial of knots with large crossing numbers in general. However, special classes of knots may allow a fast computation of some of their invariants. For example, Murakami et al. [9] recently proved that the Jones polynomial of any 2-bridge knot or link of crossing number n can be computed in $O(n^2 \ln n)$ time.

The authors greatly extended the above result to the following theorem.

Theorem 5.1 *[4] The Jones polynomial of any Conway algebraic link diagram with n crossings can be computed in $O(n^2)$ time. Consequently, the Jones polynomial of any Montesinos link or any two-bridge knot or link of crossing number n can be computed in $O(n^2)$ time.*

It is well known that the Jones polynomial $J(t)$ of a knot $\mathcal{K}$ can be obtained from a signed version of the Tutte polynomial of the face graph obtained from a regular projection D of $\mathcal{K}$ [6, 7, 8], using the substitutions below and then multiplying by $(-A^{-3})^{w(D)}$ (where $A = t^{-1/4}$ and $w(D)$ is the writhe of the diagram D):

$$\begin{array}{llll} -A^{-3} \mapsto X_+, & -A^3 \mapsto X_- & -A^3 \mapsto Y_+, & -A^{-3} \mapsto Y_-, \\ A \mapsto x_+, & A^{-1} \mapsto x_-, & A^{-1} \mapsto y_+, & A \mapsto y_-. \end{array} \tag{1}$$

The result of the theorem then follows from Theorems 3.1 and 4.1 by examining carefully how $T(F_D)$ (and hence $J(t)$) can be obtained through the repeated variable substitutions given in (1) and Theorem 4.1. Interested reader may refer to [4] for details.

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